ON SOME INEQUALITIES FOR s-LOGARITHMICALLY CONVEX FUNCTIONS IN THE SECOND SENSE VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, we establish some new Ostrowski type inequalities for $s-$logarithmically convex functions by using Riemann-Liouville fractional integrals. Some applications of our results to P.D.F.'s are given.

1. INTRODUCTION

Let $f : I \subset [0, \infty] \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a, b]$ where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality holds (see [2]).

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right] \tag{1.1}$$

This inequality is well known in the literature as the Ostrowski inequality. For some results which generalize, improve and extend the inequality (1.1) see ([2]-[6]) and the references therein.

Let us recall some known definitions and results which we will use in this paper. The function $f : [a, b] \to \mathbb{R}$, is said to be convex, if we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

In [1], Akdemir and Tunç were introduced the class of s-logarithmically convex functions in the first and second sense as the following:

Definition 1. *A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be s-logarithmically convex in the first sense if*

$$f(\alpha x + \beta y) \leq [f(x)]^{\alpha^s} [f(y)]^{\beta^s} \tag{1.2}$$

for some $s \in (0, 1]$, where $x, y \in I$ and $\alpha^s + \beta^s = 1$.

Definition 2. *A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be s-logarithmically convex in the second sense if*

$$f(tx + (1-t)y) \leq [f(x)]^{t^s} [f(y)]^{(1-t)^s} \tag{1.3}$$

for some $s \in (0, 1]$, where $x, y \in I$ and $t \in [0, 1]$.

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Clearly, when taking $s=1$ in Definition 1 or Definition 2, then f becomes the standard logarithmically convex function on I.

The main purpose of this paper is to establish some new Ostrowski's type inequalities for $s-$ logarithmically convex functions. We also give some applications to P.D.F.'s.

2. THE NEW RESULTS

In order to prove our main results, we will use following Lemma which was used by Alomari and Darus (see [3]):

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I where $a, b \in I$, with $a < b$. Let $f' \in L[a,b]$, then the following equality holds;*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)du = (b-a)\int_0^1 p(t) f'(ta + (1-t)b)\, dt$$

for each $t \in [0,1]$, where

$$p(t) = \begin{cases} t, & t \in \left[0, \frac{b-x}{b-a}\right] \\ t-1, & t \in \left(\frac{b-x}{b-a}, 1\right] \end{cases}$$

for all $x \in [a,b]$.

Theorem 1. *Let $I \supset [0,\infty)$ be an open interval and $f : I \to (0,\infty)$ is differentiable mapping on I. If $f' \in L[a,b]$ and $|f'|$ is s-logarithmically convex functions in the first sense on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq |f'(b)|\, \Psi(\tau, s, a, b) \tag{2.1}$$

where

$$\Psi(\tau, s, a, b) = \begin{cases} (b-a)\left[\dfrac{\tau^{s\left(\frac{b-x}{b-a}\right)}\left(2s\left(\frac{b-x}{b-a}\right)-1\right)+1}{2s^2\ln\tau} + \dfrac{\tau^{s} - \tau^{s\left(\frac{b-x}{b-a}\right)}\left(2s\left(\frac{x-a}{b-a}\right)-1\right)}{2s^2\ln\tau}\right] & ,\tau < 1 \\ \dfrac{(a-x)^2 + (b-x)^2}{2(b-a)} & ,\tau = 1 \end{cases}$$

and

$$\tau = \frac{|f'(a)|}{|f'(b)|}.$$

Proof. By Lemma 1 and since $|f'|$ is $s-$logarithmically convex function in the first sense on $[a,b]$, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \\
&\leq (b-a)\left[\int_0^{\frac{b-x}{b-a}} t\left| f'(ta+(1-t)b)\right| dt + \int_{\frac{b-x}{b-a}}^1 (1-t)\left| f'(ta+(1-t)b)\right| dt \right] \\
&\leq (b-a)\left[\int_0^{\frac{b-x}{b-a}} t\left| f'(a)\right|^{t^s} \left| f'(b)\right|^{1-t^s} dt + \int_{\frac{b-x}{b-a}}^1 (1-t)\left| f'(a)\right|^{t^s} \left| f'(b)\right|^{1-t^s} dt \right].
\end{aligned}$$

If $\frac{|f'(a)|}{|f'(b)|} = 1$, it easy to see that

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \left| f'(b)\right| \left[\frac{(a-x)^2 + (b-x)^2}{2(b-a)} \right].$$

If $\frac{|f'(a)|}{|f'(b)|} < 1$, then $\left(\frac{|f'(a)|}{|f'(b)|}\right)^{t^s} \leq \left(\frac{|f'(a)|}{|f'(b)|}\right)^{st}$, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \\
&\leq (b-a)\left| f'(b)\right| \left[\frac{\left(\frac{|f'(a)|}{|f'(b)|}\right)^{s\left(\frac{b-x}{b-a}\right)} \left(2s\left(\frac{b-x}{b-a}\right) - 1\right) + 1}{2s^2 \ln\left(\frac{|f'(a)|}{|f'(b)|}\right)} \right. \\
&\left. + \frac{\left(\frac{|f'(a)|}{|f'(b)|}\right)^{s} - \left(\frac{|f'(a)|}{|f'(b)|}\right)^{s\left(\frac{b-x}{b-a}\right)} \left(2s\left(\frac{x-a}{b-a}\right) - 1\right)}{2s^2 \ln\left(\frac{|f'(a)|}{|f'(b)|}\right)} \right].
\end{aligned}$$

This completes the proof. □

Corollary 1. *If we choose $|f'| \leq M$ in (2.1), we obtain the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq M\Psi(\tau, s, a, b)$$

where

$$\Psi_1(\tau,s,a,b)=\begin{cases}(b-a)\left[\frac{M^{s\left(\frac{b-x}{b-a}\right)}\left(2s\left(\frac{b-x}{b-a}\right)-1\right)+1}{2s^2\ln M}+\frac{M^s-M^{s\left(\frac{b-x}{b-a}\right)}\left(2s\left(\frac{x-a}{b-a}\right)-1\right)}{2s^2\ln M}\right] & ,M<1\\ \frac{(a-x)^2+(b-x)^2}{2(b-a)} & ,M=1\end{cases}.$$

Corollary 2. *If we choose* $x=\frac{a+b}{2}$ *in (2.1), we obtain the inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\left|f'(b)\right|\Psi(\tau,s,a,b)$$

where

$$\Psi(\tau,s,a,b)=\begin{cases}(b-a)\left[\frac{\tau^{\left(\frac{s(b-a)}{2}\right)}(s(b-a)-1)+1}{2s^2\ln\tau}+\frac{\tau^s-\tau^{\frac{s(b-a)}{2}}(s(b-a)-1)}{2s^2\ln\tau}\right] & ,\tau<1\\ \frac{b-a}{4} & ,\tau=1\end{cases}$$

and

$$\tau=\frac{|f'(a)|}{|f'(b)|}.$$

Corollary 3. *If we choose* $s=1$ *in (2.1), we obtain the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\left|f'(b)\right|\Psi(\tau,1,a,b)$$

where

$$\Psi(\tau,1,a,b)=\begin{cases}(b-a)\left[\frac{\tau^{\left(\frac{b-x}{b-a}\right)}\left(2\left(\frac{b-x}{b-a}\right)-1\right)+1}{2\ln\tau}+\frac{\tau-\tau^{\left(\frac{b-x}{b-a}\right)}\left(2\left(\frac{x-a}{b-a}\right)-1\right)}{2\ln\tau}\right] & ,\tau<1\\ \frac{(a-x)^2+(b-x)^2}{2(b-a)} & ,\tau=1\end{cases}$$

and

$$\tau=\frac{|f'(a)|}{|f'(b)|}.$$

Theorem 2. *Let* $I\supset[0,\infty)$ *be an open interval and* $f:I\to(0,\infty)$ *is differentiable mapping on* I. *If* $f'\in L[a,b]$ *and* $|f'|^q$ *is* $s-$*logarithmically convex functions in the first sense on* $[a,b]$ *for some fixed* $s\in(0,1]$, *then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{|f'(b)|}{(p+1)^{\frac{1}{p}}}\Psi(\tau,s,a,b) \tag{2.2}$$

where

$$\Psi(\tau,s,a,b)=\begin{cases}\frac{1}{(b-a)^{\frac{1}{p}}}\left[(b-x)^{\frac{p+1}{p}}\left(\frac{\tau^{sq\left(\frac{b-x}{b-a}\right)}-1}{sq\ln\tau}\right)^{\frac{1}{q}}+(x-a)^{\frac{p+1}{p}}\left(\frac{\tau^{sq}-\tau^{sq\left(\frac{b-x}{b-a}\right)}}{sq\ln\tau}\right)^{\frac{1}{q}}\right] & ,\tau<1\\ \left[\frac{(b-x)^2+(x-a)^2}{b-a}\right] & ,\tau=1\end{cases}$$

and

$$\tau = \frac{|f'(a)|}{|f'(b)|}.$$

for $q > 1$ *and* $p^{-1} + q^{-1} = 1$.

Proof. From Lemma 1 and by using the Hölder integral inequality, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \\
&\leq (b-a)\left[\left(\int_0^{\frac{b-x}{b-a}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Since $|f'|$ is $s-$logarithmically convex function in the first sense, we can write

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \\
&\leq (b-a)|f'(b)|\left[\left(\int_0^{\frac{b-x}{b-a}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} \left(\frac{|f'(a)|}{|f'(b)|} \right)^{qt^s} dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 \left(\frac{|f'(a)|}{|f'(b)|} \right)^{qt^s} dt \right)^{\frac{1}{q}} \right].
\end{aligned}$$

If $\frac{|f'(a)|}{|f'(b)|} = 1$, then we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{|f'(b)|}{(p+1)^{\frac{1}{p}}} \left[\frac{(b-x)^2 + (x-a)^2}{b-a} \right].$$

On the other hand, if $\frac{|f'(a)|}{|f'(b)|}<1$, then $\left(\frac{|f'(a)|}{|f'(b)|}\right)^{qt^s} \leq \left(\frac{|f'(a)|}{|f'(b)|}\right)^{sqt}$, thereby

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
\leq\ & (b-a)\left|f'(b)\right|\left[\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left(\frac{|f'(a)|}{|f'(b)|}\right)^{sqt} dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left(\frac{|f'(a)|}{|f'(b)|}\right)^{sqt} dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

By computing the above integrals the proof is completed. □

Corollary 4. *If we choose $|f'|\leq M$ in (2.2), we obtain the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\Psi(\tau,s,a,b)$$

where

$$\Psi(\tau,s,a,b)=\begin{cases}\frac{1}{(b-a)^{\frac{1}{p}}}\left[(b-x)^{\frac{p+1}{p}}\left(\frac{M^{sq\left(\frac{b-x}{b-a}\right)}-1}{sq\ln M}\right)^{\frac{1}{q}}+(x-a)^{\frac{p+1}{p}}\left(\frac{M^{sq}-M^{sq\left(\frac{b-x}{b-a}\right)}}{sq\ln M}\right)^{\frac{1}{q}}\right] & ,M<1 \\ \left[\frac{(b-x)^2+(x-a)^2}{b-a}\right] & ,M=1\end{cases}.$$

Corollary 5. *If we choose $x=\frac{a+b}{2}$ in (2.2), we obtain the inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \left(\frac{b-a}{2}\right)\frac{|f'(b)|}{(p+1)^{\frac{1}{p}}}\Psi(\tau,s,a,b)$$

where

$$\Psi(\tau,s,a,b)=\begin{cases}\left(\frac{\tau^{sq\left(\frac{b-a}{2}\right)}-1}{sq\ln\tau}\right)^{\frac{1}{q}}+\left(\frac{\tau^{sq}-\tau^{sq\left(\frac{b-a}{2}\right)}}{sq\ln\tau}\right)^{\frac{1}{q}} & ,\tau<1 \\ \frac{1}{2} & ,\tau=1\end{cases}$$

and

$$\tau=\frac{|f'(a)|}{|f'(b)|}.$$

3. APPLICATIONS FOR P.D.F's

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x)=\Pr(X\leq x)=\int_a^b f(t)dt$.

Theorem 3. *Under the assumptions of Theorem 1, we have the inequality;*

$$\left|\Pr(X \leq x) - \frac{1}{b-a}\left(b - E(x)\right)\right| \leq \left|f'\left(b\right)\right| \Psi\left(\tau, s, a, b\right)$$

where $E(x)$ *is the expectation of* X *and* $\Psi\left(\tau, s, a, b\right)$ *as defined in Theorem 1.*

Proof. The proof is immediate follows from the fact that;

$$E(x) = \int_a^b t dF(t) = b - \int_a^b F\left(t\right) dt.$$

□

Theorem 4. *Under the assumptions of Theorem 2, we have the inequality;*

$$\left|\Pr(X \leq x) - \frac{1}{b-a}\left(b - E(x)\right)\right| \leq \frac{\left|f'\left(b\right)\right|}{\left(p+1\right)^{\frac{1}{p}}} \Psi\left(\tau, s, a, b\right)$$

where $E(x)$ *is the expectation of* X *and* $\Psi\left(\tau, s, a, b\right)$ *as defined in Theorem 2.*

Proof. Likewise the proof of the previous theorem, by using the fact that;

$$E(x) = \int_a^b t dF(t) = b - \int_a^b F\left(t\right) dt$$

the proof is completed. □

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